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Fourth Order Compact Method for One Dimensional Inhomogeneous Telegraph Equation of $O(h^4, k^3)$

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Abstract

Many boundary value problems that arise in real life situation defy analytical solutions; hence numerical techniques are the best source for finding the solution of such equations. In this study Finite difference Method (FDM) and Fourth Order Compact Method (FOCM) are presented for the solutions of well known one dimensional Inhomogeneous Telegraph equation and then its validity and applicability is checked through applications. The results obtained are compared with the exact solutions for these applications. We used Fortran 90 for the calculations of the numerical results and Mat lab for the graphical comparison.

Key Words: Inhomogeneous Telegraph Equation; FOC Method; FD Method

1. Introduction

A general fourth Order differencing scheme proposed by H.O. Kreiss of Uppsala University is developed and tested to three viscous test problems to verify the correctness and applicability of the method. The method is a typical since only three nodes are required to attain the desired fourth order precision. This is proficient by a differencing procedure, which considers the function and all required derivatives as unknowns. The relations for these derivatives give up simple tridiagonal equations, which can be solved effortlessly. In (ORSZAG; 1974) a compact formula was mentioned. This method was used in that style by Ciment and Leventhal (1978) for hyperbolic problems. Abdul Majid Wazwaz [3] explained different techniques to solve a variety of PDEs. In Numerical Analysis by Richard L. Burden [4] explained in detail the finite difference method for different partial differential equations. Ozair [5,6,7] used compact methods and compare their results with finite difference scheme results. Consider the 2nd order 1D linear hyperbolic equation.

$$\alpha \frac{\partial^2 u(x,t)}{\partial t^2} + \beta \frac{\partial u(x,t)}{\partial t} + \gamma u(x,t) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t) \tag{1}$$

with the following initial conditions

$$u(x.0) = f(x) \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t} = g(x) \tag{3}$$

and with the boundary conditions

$$u(0,t) = 0 \tag{4}$$

$$u(l,t) = 0 \tag{5}$$

for $0 \le x \le l, \quad t > 0$

Eq. (1) is referred to as the second order Telegraph Equation with constant coefficients. In eq. (1), x is distance and t is time. For $\alpha > 0, \beta = 0$ eq. (1) represents a damped wave equation and for $\alpha, \beta, \gamma, c^2$ are non negative integers then it is called telegraph equation.

2. Finite Difference Scheme

To set up the finite difference scheme for eq. (1), select an integer m and the values of t from 0 to ∞ then the mesh points (x_i, t_n) are

$$x_t = t\Delta x = th \qquad \text{for} \quad t = 0, 1, 2, 3, \ldots \text{m}$$
$$t_n = n\Delta t = nk \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

At any interior mesh points $x_i, t_n)$, then the Hyperbolic Homogeneous Telegraph eq. (1) becomes

$$\alpha\frac{\partial^2 u(x_1,t_n)}{\partial t^2}+\beta\frac{\partial u(x_i,t_n)}{\partial t}+\gamma u(x_i,t_n)= c^2\frac{\partial^2 u(x_i,t_n)}{\partial x^2}+p(x_i,t_n) \qquad (6)$$

The method is obtained using the central difference approximation for the 1st and 2nd order partial derivatives.

So that (6) becomes

$$\frac{\alpha}{(\Delta t)^2}(u_i^{n+1}-2u_i^n+2u_i^{n-1})-\frac{\alpha(\Delta t)^2}{12}\frac{\partial^4 u(x_i,\mu_n)}{\partial t^4}$$

$$+\frac{\beta}{2(\Delta t)}(u_i^{n+1}-u_i^{n-1})$$

$$+\frac{\beta(\Delta t)^2}{6}\frac{\partial^3 u(x_i,\mu_n)}{\partial t^3}+\gamma u_i^n$$

$$=\frac{c^2}{(\Delta x)^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n$$

$$-\frac{c^2(\Delta x)^2}{12}\frac{\partial^4 u(\xi_i,t_n)}{\partial x^4}$$

Where $\xi_i = (x_i, x_{i+t})$

Neglecting the truncation error leads to the difference equation.

$$\frac{\alpha}{(\Delta t)^2}(u_i^{n+1}-2u_i^n+u_i^{n-1})+\frac{\beta}{2(\Delta t)}(u_i^{n+1}-u_i^{n-1})$$

$$=\frac{c^2}{(\Delta x)^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n)$$

$$-\frac{c^2}{(\Delta x)^2}(u_{i+1}^n+u_{i-1}^n)=\left(\frac{\alpha}{(\Delta t)^2}+\frac{\beta}{2(\Delta t)}\right)u_i^{n+1}+$$

$$\left(\gamma-\frac{2a}{(\Delta t)^2}+\frac{2c^2}{(\Delta t)^2}\right)u_i^n+\left(\frac{\alpha}{(\Delta t)^2}+\frac{\beta}{2(\Delta t)}\right)u_i^{n-1}$$

Taking

$$\left(\frac{\alpha}{(\Delta t)^2}+\frac{\beta}{2(\Delta t)}\right)=\lambda_1.\left(\gamma-\frac{2a}{(\Delta t)^2}+\frac{2c^2}{(\Delta t)^2}\right)=\lambda_2$$

and $\left(\frac{\alpha}{(\Delta t)^2}+\frac{\beta}{2(\Delta t)}\right)=\lambda_3$

So

$$\frac{c^2}{(\Delta x)^2}(u_{i+1}^n+u_{i-1}^n)+p_i^n$$

$$=\lambda_1 u_i^{n+1}+\lambda_2 u_i^n+\lambda_3 u_i^{n-1}$$

$$\lambda_1 u_i^{n+1}=\frac{c^2}{(\Delta x)^2}(u_{i+1}^n+u_{i-1}^n)-\lambda_2 u_i^n-\lambda_1 u_i^{n-1}+p_i^n$$

$$u_i^{n+1}=\frac{c^2}{\lambda_1 \Delta x)^2}(u_{i+1}^n+u_{i-1}^n)-\frac{\lambda_2}{\lambda_1}u_i^n-\frac{\lambda_3}{\lambda_1}u_i^{n-1}$$

$$\frac{1}{\lambda_1}p_i^n+$$

By letting $\frac{c^2}{\lambda_1(\Delta x)^2}=\Lambda, \frac{-\lambda_2}{\lambda_1}=\psi, \quad \frac{-\lambda_3}{\lambda_1}=\Phi,$ and $\frac{1}{\lambda_1}=\Omega$.

So

$$u_i^{n+1}=\Lambda(u_{i+1}^n+u_{i-1}^n)+\psi u_i^n+\Phi u_i^{n-1}+\Omega p_i^n$$

$$u_i^{n+1}=\psi(u_i^n+\Lambda u_{i+1}^n+\Lambda u_{i-1}^n+\Phi u_i^{n-1}+\Omega p_i^n \qquad (7)$$

This equation holds for each I =1, 2, …, $(m-1)$. The boundary conditions give

$$u_0^n=u_m^n=0 \qquad (8)$$

for each n = 1, 2, ….

And the initial condition implies that

$$u_1^0=f(x_i) \qquad (9)$$

for i = 1, 2, …. $(m-1)$.

Writing in matrix form for i = 1, 2, …. $(m-1)$, we have

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{m-1}^{n+1} \end{bmatrix}=\begin{bmatrix} \psi & \Lambda & 0 & & 0 \\ \Lambda & \psi & \Lambda & & \\ 0 & \Lambda & \ddots & \ddots & 0 \\ & & \ddots & \psi & \Lambda \\ 0 & & 0 & \Lambda & \psi \end{bmatrix}\begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{m-1}^n \end{bmatrix}+\Phi\begin{bmatrix} u_1^{n-1} \\ u_2^n \\ \vdots \\ \vdots \\ u_{m-1}^{n-1} \end{bmatrix}+$$

$$\Omega\begin{bmatrix} p_1^{n+1} \\ p_2^n \\ \vdots \\ \vdots \\ p_{m-1}^{n+1} \end{bmatrix} \qquad (10)$$

Equations (7) and (8) imply that the $(n+1)^{th}$ time steps requires values from the $(n)^{th}$ and $(n-1)^{th}$ time steps. This produces a minor starting problem since values of n = 1 which is needed, in equation (7) to compute u_1^2 must be obtained from the initial value condition.

$$u_t\,|_1^0=g(x_i), \qquad 0\le x\le l$$

A better approximation $u_t|_1^0$ can be obtained rather easily, particularly when the second derivative of 'f' at 'x_i' can be determined.

Consider the Taylor Series

$$u_i^{n+1} = u_i^n + k u_t|_i^m + \frac{k^2}{2} k u_{tt}|_i^n + \frac{k^3}{6} u_{ttt}|_i^n + \frac{k^4}{24} u_{tttt}|_i^n + \frac{k^5}{120} u_{ttttt}|_i^n + \ldots$$

$$\frac{u_i^{n+1} - u_i^n}{k} = u_t|_i^n + \frac{k}{2} u_{tt}|_i^n + \frac{k^2}{6} u_i^{(3)n}(x_i, \mu_n)$$

For $n = 0$, we have

$$\frac{u_i^1 - u_i^0}{k} = u_t|_i^0 + \frac{k}{2} u_{tt}|_i^0 + \frac{k^2}{6} u_i^{(3)0}(x_i, \mu_n) \tag{11}$$

for some μ_n in $(0, t_1)$ and suppose the inhomogeneous telegraph equation also holds on the initial line. That is

$$u_{tt}|_i^0 = \frac{c^2}{\alpha} f''(x_i) - \frac{\beta}{\alpha} u_t)|_i^0 - \frac{\gamma}{\alpha} u_i^0 + \frac{1}{6} p_i^0$$

Substituting this value in eq.(11), we get

$$\frac{u_i^1 - u_i^0}{k} = u_t|_i^0 + \frac{k}{2}\left(\frac{c^2}{\alpha} f''(x_i) - \frac{\beta}{\alpha} u_t|_i^0 - \frac{\gamma}{\alpha} u_i^0 + \frac{1}{\alpha} p_i^0\right) + \frac{k^2}{6} u_i^{(3)0}(x_i, \mu_n)$$

but

$$u_t|_i^0 = g(x_i)$$

So on simplifying we get

$$u_i^1 = \frac{k^2 c^2}{2\alpha} f''(x_i) + \left(k - \frac{\beta k^2}{2\alpha}\right) g(x_i) + \left(1 - \frac{\gamma k^2}{2\alpha}\right) u_i^0 + \frac{k^2}{2\alpha} p_i^0$$

This is an approximation with local truncation error $O(k^3)$ for each $i = 1, 2, \ldots, m-1$.

Now from the difference equation

$$f''(x_i) + \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$u_i^1 = \frac{k^2 c^2}{2\alpha}\left(\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}\right) + \left(k - \frac{\beta k^2}{2a}\right) g(x_i) + \left(1 - \frac{\gamma k^2}{2a}\right) u_i^0 + \frac{k^2}{2a} p_i^0$$

$$u_i^1 = \frac{k^2 c^2}{2\alpha h^2} + f(x_{i-1})\left(k - \frac{\beta k^2}{2\alpha}\right) g(x_i) + \left(1 - \frac{\gamma k^2}{2\alpha}\right) u_i^0 - \frac{k^2 c^2}{\alpha h^2} f(x_i) + \frac{k^2}{2\alpha} p_i^0$$

But $u_i^0 = f(x_i)$ and letting $\lambda^2 = \frac{k^2 c^2}{h^2}$, we have

$$u_i^1 = \frac{\lambda^2}{2\alpha}\left(f(x_{i+1})\right) + f(x_{i-1}) + \left(k - \frac{\beta k^2}{2\alpha}\right) g(x_i) + \left(1 - \frac{\gamma k^2}{2\alpha} - \frac{\lambda^2}{\alpha}\right) f(x_i) + \frac{k^2}{2\alpha} p_i^0 \tag{12}$$

for each $i = 1, 2, \ldots (m-1)$.

3. Compact Scheme for Inhomogeneous Telegraph Equation:

To derive this method for the second order linear hyperbolic telegraph eq. (1), with > 0, $\beta > 0$, $\gamma = 0$, $c^2 > 0$, $f(x$ and $g(x)$ are given functions. This Compact method approximates eq. (1) by two difference equations of fourth order using only three grid points say x_{i-1}, x_i and x_{i+1}. Let us denote 1st and 2nd derivatives of $u(x, t)$ with respect to 'x' by F, S respectively.

$$u_x(x,t) = F \tag{13}$$
$$u_{xx}(x,t) = S$$

We shall first develop a link between the values of F and u. Since $F = u_x$, it is clear that

$$u_{i+1}^n = u_{i-1}^n + \int_{i-1}^{i+1} F(\xi, l)\, d\xi$$

Approximating this integral by Simpson's Rule and reorganizing we get

$$u_{i+1}^n = u_{i-1}^n + \frac{h}{3}(F_{i-1}^n + F_i^n + F_{i+1}^n) + \frac{h^5}{90} \frac{\partial^4 F(\xi, l)}{\partial x^4} \text{u}$$

Thus to fourth order, we have

$$(F_{i-1}^n + 4F_i^n + F_{i+1}^n) + \frac{h}{3}(u_{i-1}^n - u_{i+1}^n) = 0 \text{ u} \tag{14}$$

So we have a relationship between u and F. This is the 1st difference equation.

In order to obtain the 2nd equation, we start by evaluating (1) at the mid point 'i'. Then eq. (1) becomes

$$\alpha u_{tt}|_1^n + \beta u_t|_1^n + \gamma u Z|_i^n = c^2 S|_i^n + p_i^n \tag{15}$$

We now need the term for $S|_i^n$. If we articulate $u|_{i+1}^n$ and $u|_{i-1}^n$ in Taylor series about the point (i,n) and adding the result we get

$$u_{i+1}^n + u_{i-1}^n = 2u_i^n + h^2 S|_i^n + \tfrac{h^4}{12} u_{xxxx}|_i^n + \tfrac{h^6}{360} u_{xxxxxx}(\xi,l)|_i^n \text{ s} \tag{16}$$

where we have replaced u_{xx} with $S|_i^n$. If we carry out the same procedure for F then we have

$$F_{i+1}^n - F_{i-1}^n = 2hS|_i^n + \tfrac{h^3}{3} u_{xxx}|_i^n + \tfrac{h^5}{60} u_{xxxxx}(\xi,l)|_i^n \tag{17}$$

We now eliminate $u_{xxxx}|_i^n$ from these two equations and after rearranging, we get the following expression for $S|_i^n$, $S|_{i-1}^n$ and $S|_{i+1}^n$

$$S|_i^n = \tfrac{2}{h^2}(u_{i+1}^n + u_{i-1}^n - 2u_i^n) - \tfrac{1}{2h}(F_{i+1}^n - F_{i-1}^n) + \tfrac{h^4}{360} u_{xxxxxx}(\xi,l)|_i^n$$

By a similar procedure we get the following expressions for $S|_{i-1}^n$ and $S|_{i+1}^n$

$$S|_{i-1}^n = \tfrac{1}{2h^2}(7u_{i+1}^n - 23u_{i-1}^n + 16u_i^n) - \tfrac{1}{h}(F_{i+1}^n + 6F_{i-1}^n + 8F_i^n) + \tfrac{h^4}{90} u_{xxxxxx}(\xi,l)|_i^n$$

and

$$S|_{i+1}^n = \tfrac{1}{2h^2}(7u_{i-1}^n - 23u_{i+1}^n + 16u_i^n) + \tfrac{1}{h}(F_{i-1}^n + 6F_{i+1}^n + 8F_i^n) + \tfrac{h^4}{90} u_{xxxxxx}(\xi,l)|_i^n$$

We now surrogate the expression for $S|_i^n$ into (15) and reorganize to get the following 2nd difference equation of fourth order.

$$\alpha u_{tt}|_1^n + \beta u_t|_1^n = \tfrac{2c^2}{h^2}(u_{i+1}^n + u_{i-1}^n)\left(\gamma + \tfrac{4c^2}{h^2}\right)u_i^n - \tfrac{c^2}{2h}(F_{i+1}^n - F_{i-1}^n) + p_i^n \tag{18}$$

We have now replaced (1) by two difference equations (14) and (18). Now we have to look at the boundaries. Let us first deem the left boundary condition i.e., at $x = 0$ and denotes the points $x = 0, h, 2h$ by 0, 1, 2. The 1st difference equation we obtain from the boundary condition is

$$u_0^n = 0 \tag{19}$$

In order to get the 2nd equation, we begin with the differential equation at the point 0 and 1.

$$c^2 S|_0^n + p_0^n = a u_{tt}|_0^n + \beta u_t|_0^n + \gamma u|_0^n \tag{20}$$

$$c^2 S|_1^n + p_1^n = a u_{tt}|_1^n + \beta u_t|_1^n + \gamma u|_1^n \tag{21}$$

From the above equations of $S|_1^n$, $S|_{1-1}^n$ and $S|_{1+1}^n$, we have the following expressions for $S|_0^n$ and $S|_1^n$.

$$S|_0^n = \tfrac{1}{2h^2}(-23u_0^n + 16u_1^n + 7u_2^n) - \tfrac{1}{h}(6F_0^n + 8F_1^n + F_2^n) \tag{22}$$

$$S|_1^n = \tfrac{2}{h^2}(u_0^n - 2u_1^n + u_2^n) - \tfrac{1}{2h}(F_2^n - F_0^n) \tag{23}$$

Finally we have from (14)

$$(F_0^n + F_1^n + F_2^n) + \tfrac{h}{3}(u_0^n - u_2^n) = 0 \tag{24}$$

So we have five equations (20) to (24). If we eliminate u_2^n, $S|_0^n$, $S|_1^n$ and $F|_2^n$ from these five equations, we get the 2nd difference equation, suitable at $x = 0$.

$$\left(\tfrac{12c^2}{h^2} + \gamma\right)u_0^n - \left(\tfrac{12c^2}{h^2} + \gamma\right)u_1^n + \tfrac{6c^2}{h} F_0^n + \tfrac{6c^2}{h} F_1^n + (p_1^n - p_0^n) = \alpha(u_{tt}|_1^n - u_{tt}|_0^n) + \beta(u_t|_1^n - u_t|_0^n) \tag{25}$$

In a similar manner, we can derive the following difference equation for u and F at $x = m$, i.e. at the right boundary point.

$$u_m^n = 0 \tag{26}$$

$$\left(\tfrac{12c^2}{h^2} + \gamma\right)u_{m-1}^n - \left(\tfrac{12c^2}{h^2} + \gamma\right)u_m^n + \tfrac{6c^2}{h} F_{m-1}^n + \tfrac{6c^2}{h} F_m^n + (p_m^n - p_{m-1}^n) = \alpha(u_{tt}|_m^n - u_{tt}|_{m-1}^n) + \beta(u_t|_m^n - u_t|_{m-1}^n) \tag{27}$$

Thus for ach point, we have two difference equations. If we write them all together, we have the following Fourth Order Compact Scheme for u_{xx}.

$$u_0^n = 0$$

$$\left(\tfrac{12c^2}{h^2} + \gamma\right)u_0^n - \left(\tfrac{12c^2}{h^2} + \gamma\right)u_1^n + \tfrac{6c^2}{h} F_0^n + \tfrac{6c^2}{h} F_1^n + (p_1^n - p_0^n)$$

$$=\alpha(u_{tt}|_1^n - u_{tt}|_0^n) + \beta(u_t|_1^n - u_t|_0^n)$$

$$\frac{2c^2}{h^2}(u_{i+1}^n + u_{i-1}^n) - \left(\gamma + \frac{4c^2}{h^2}\right)u_i^n$$

$$-\frac{c^2}{2h}(F_{i+1}^n - F_{i-1}^n) + p_i^n$$

$$=\alpha(u_{tt}|_1^n + \beta(u_t|_1^n$$

$$u_m^n = 0$$

$$\left(\frac{12c^2}{h^2} + \gamma\right)u_{m-1}^n = \left(\frac{12c^2}{h^2} + \gamma\right)u_m^n + \frac{6c^2}{h}F_{m-1}^n$$

$$+\frac{6c^2}{h}F_m^n + (p_m^n - p_{m-1}^n)$$

$$=\alpha(u_{tt}|_m^n - u_{tt}|_{m-1}^n)$$

$$+\beta(u_t|_m^n - u_t|_{m-1}^n)$$

The superscript n is used to denote the time grid lines.

Difference scheme using compact scheme for u_{xx} and central difference scheme for u_{tt} and u_t.

$$u_{tt}|_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{k^2} + O(k^2)$$

and

$$u_t|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2k} + O(k^2)$$

We have from eqs. (19), (25), (24),(18), (26) and (27) as below:

$$u_0^n = 0$$

$$\left(\frac{\alpha}{k^2} + \frac{\beta}{2k}\right)u_i^{n+1} - \frac{6c^2}{h}F_0^n - \frac{6c^2}{h}F_1^n = \left(\frac{2\alpha}{k^2} - \frac{12c^2}{h^2} - \gamma\right)u_i^n + \left(\frac{\beta}{2k} - \frac{\alpha}{k^2}\right)u_i^{n-1} + (p_i^n - p_o^n) \qquad (28)$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n = \frac{h}{3}(u_{i+1}^n - u_{i-1}^n) \qquad (29)$$

$$\left(\frac{\alpha}{k^2} + \frac{\beta}{2k}\right)u_i^{n+1} + \frac{c^2}{2h}F_{i+1}^n - \frac{c^2}{2h}F_{i-1}^n = \frac{2c^2}{h^2} * u_{i+1}^n + u_{i-1}^n + \left(\frac{2\alpha}{k^2} - \frac{4c^2}{h^2} - \gamma\right)u_i^n + \left(\frac{\beta}{2k} - \frac{\alpha}{k^2}\right)u_i^{n-1} + p_i^n \qquad (30)$$

$$u_m^n = 0$$

$$\left(\frac{\alpha}{k^2} + \frac{\beta}{2k}\right)u_{m-1}^{n+1} + \frac{6c^2}{h}F_{m-1}^n - \frac{6c^2}{h}F_m^n = \left(\frac{2\alpha}{k^2} - \frac{12c^2}{h^2} - \gamma\right)u_{m-1}^n + \left(\frac{\beta}{2k} - \frac{\alpha}{k^2}\right)u_{m-1}^{n-1} + (p_{m-1}^n - p_m^n) \qquad (31)$$

Now for finding u_i^1 for the next time level, we use the initial condition

$$u_t{}_1^0 = g(x_i) \qquad 0 \le x \le l$$

Which can be approximated into the form by using Taylor's series and finite differences as given in eq.(12). The Fourth Order Scheme can be expressed in matrix form.

4. Test Problem

Consider the inhomogeneous telegraph equation $u_{xx} + (1+\pi^2 e^{-t}\sin\pi x = u_{tt} + u_t + u$ in the interval $0 < x < 1$. The boundary conditions are

$$u(0,t) = u(1,t) = 0$$

and the initial conditions are

$$u(x,0) = \sin\pi x \text{ and } u_t(x,0) = -\sin\pi x, \quad 0 \le x \le 1$$

The Exact Solution is $u(x,t) = e^{-t}\sin\pi x$.

By using equation (12) we have the values for

Solution:

$$u_1^1,\ t = 1, 2, \ldots, 9$$

$u_1^1 = 0.305943525$, $u_2^1 = 0.581939167$
$u_3^1 = 0.800970548$, $u_4^1 = 0.941597351$
$u_5^1 = 0.990054045$, $u_6^1 = 0.941597351$
$u_7^1 = 0.800970548$, $u_8^1 = 0.581939167$
$u_9^1 = 0.305943525$

Now using equation (10) we have following nine finite difference method values i.e.,

$$u_1^{n+1}, n=1,\ t = 1, 2, \ldots, 9$$

$u_1^2 = 0.302903100$, $u_2^2 = 0.576155935$
$u_3^2 = 0.793010611$, $u_4^2 = 0.932239884$
$u_5^2 = 0.980215021$, $u_6^2 = 0.932239884$
$u_7^2 = 0.793010611$, $u_8^2 = 0.576155935$

To find the values of the fourth order compact method we will use equations (28-31) and following nine values are obtained i.e.,

$$u_1^{n+1}, n=1,\ t = 1, 2, \ldots, 9$$

$u_1^2 = 0.302900920$, $u_2^2 = 0.576151156$
$u_3^2 = 0.793004162$, $u_4^2 = 0.932232272$
$u_5^2 = 0.980207027$, $u_6^2 = 0.932232272$
$u_7^2 = 0.793004162$, $u_8^2 = 0.576151156$

Comparison of the Numerical Results of FDM and FOCM

Table 1: Finite Difference Method at t=0.02

x_i	FDM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.302903100	0.302898048	0.000005052
0.200000000	0.576155935	0.576146325	0.000009610
0.300000000	0.793010611	0.792997385	0.000013226
0.400000000	0.932239884	0.932224336	0.000015548
0.500000000	0.980215021	0.980198673	0.000016348
0.600000000	0.932239884	0.932224336	0.000015480
0.700000000	0.793010611	0.792997385	0.000013226
0.800000000	0.576155935	0.576146326	0.000009610
0.900000000	0.302903100	0.302898048	0.000005052
1.000000000	0.000000000	0.000000000	0.000000000

Table 2 Fourth Order Compact Method at t = 0.02

x_i	FOCM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.302900920	0.302898048	0.000002087
0.200000000	0.576151156	0.576146325	0.000004831
0.300000000	0.793004162	0.792997385	0.000006777
0.400000000	0.932232272	0.932224336	0.000007936
0.500000000	0.980207027	0.980198673	0.000008354
0.600000000	0.932232272	0.932224336	0.000007936
0.700000000	0.793004162	0.792997385	0.000006777
0.800000000	0.576151156	0.576146325	0.000008831
0.900000000	0.302900920	0.302898048	0.000002087
1.000000000	0.000000000	0.000000000	0.000000000

For graph see Figure 1

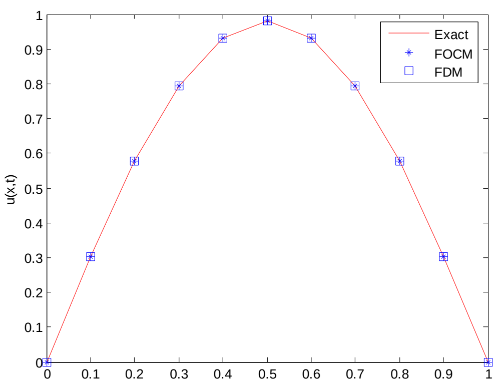

Figure 1

5. Conclusion

In this paper, numerical solutions of the one dimensional linear inhomogeneous telegraph equation are derived using Finite Difference Method and Fourth Order Compact Method. Fourth Order Compact Method is known to be a powerful device for solving functional equations. From the solutions of inhomogeneous telegraph equation, we note that the fourth order compact method with $O(h^4, k^3)$, which also uses only three nodes, gives better results than the usual second order method.

6. References

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